ON BASES, FINITE DIMENSIONAL DECOMPOSITIONS AND WEAKER STRUCTURES IN BANACH SPACES

BY

W. B. JOHNSON, H. P. ROSENTHAL* AND M. ZIPPIN*

ABSTRACT

This is an investigation of the connections between bases and weaker structures in Banach spaces and their duals. It is proved, e.g., that X has a basis if X^* does, and that if X has a basis, then X^* has a basis provided that X^* is separable and satisfies Grothendieck's approximation property; analogous results are obtained concerning π-structures and finite dimensional Schauder decompositions. The basic results are then applied to show that every separable $\mathscr{L}_p$ space has a basis.

1. Introduction. Very little is known about the geometry of a general Banach space, its subspaces and its projections. However, the common Banach spaces usually have a family $\{E_\alpha\}$, directed by inclusion, of "nice" finite dimensional subspaces, which span the whole space. We will call such a family a *structure*. The effect of the structure on the properties of the space obviously depends on how nice the subspaces E_α are. In several instances (see e.g. [7], [9] and [10]), where the subspaces E_α satisfy some strong properties, interesting theories are developed, thus demonstrating that a Banach space with a structure is easier to handle. In the sequel we will restrict ourselves to four types of structure determined by the following properties:

(1) The bounded approximation property (b.a.p., in short) (cf. [3] p. 182). Let $\lambda \geqq 1$. A Banach space X is said to have the λ-metric approximation property (λ-m.a.p. in short) if for every finite dimensional subspace $E \subset X$ and every $\varepsilon > 0$ there is an operator T with finite dimensional range on X such that $\|T\| \leqq \lambda$ and $\|Tx-x\| \leqq \varepsilon \|x\|$ for all $x \in E$. The space X is said to have the bounded approximation property if it has the λ-m.a.p. for some $\lambda \geqq 1$.

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(2) *The π property* (cf. [4] and [7] p. 25): Let $\lambda \geqq 1$. A Banach space X is called a π_λ space if $X = \overline{\bigcup_{\alpha \in A} E_\alpha}$, where $\{E_\alpha\}_{\alpha \in A}$ is a set, directed by inclusion, of finite dimensional subspaces of X, such that for every $\alpha \in A$ there is a projection T_α from X onto E_α with $\| T_\alpha \| \leqq \lambda$. The space X is called a π space if it is a π_λ space for some $\lambda \geqq 1$.

(3) *The finite dimensional decomposition property* (f.d.d.p., in short): A Banach space X is said to have the f.d.d.p. if there is a sequence $\{F_n\}$ of finite dimensional subspaces of X such that each $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} P_n x$ with $P_n x \in F_n$ for all n. The sequence $\{F_n\}$ is called a finite dimensional decomposition (f.d.d., in short) for X. It is known that the functions $P_n x$ are bounded linear projections on X and that $P_n P_k = \delta_{n,k} P_k$ for all n and k. Moreover, for each n, the operator $Q_n = \sum_{k=1}^{n} P_k$ is a projection from X onto $E_n = \operatorname{span}\{\cup_{k=1}^{n} F_k\}$, $Q_n \to I$ strongly and $Q_n Q_k = Q_k Q_n = Q_{min(k,n)}$ for all k and n. We will refer to the Q_n as the projections which determine the decomposition, or, the natural projections of the decomposition.

(4) *The basis property.* A sequence $\{x_n\}$ of elements of a Banach space X is called a *basis* if each $x \in X$ has a unique representation $x = \sum_1^{\infty} a_i x_i$ where $\{a_i\}$ are scalars. A space X is said to have the basis property if it has a basis. The projections $\{Q_n\}$ defined by $Q_n(\sum_1^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$ are linear and bounded; they will be called the natural projections of the basis.

Each of the first two properties has several equivalent definitions which may be more convenient in various situations. Using Lemma 2.4 below, one can easily prove the following

PROPOSITION 1.1. *Let X be a Banach space. Then* (a), (b) *and* (c) *below are equivalent:*

(a) *X has the b.a.p.*

(b) *There is a uniformly bounded net $\{T_\alpha\}$ of operators with finite dimensional ranges which tends strongly to the identity on X.*

(c) *There is a $\lambda \geqq 1$ satisfying the following property: for every finite dimensional subspace $E \subset X$ there is an operator T with finite dimensional range on X such that $\| T \| \leqq \lambda$ and $Tx = x$ for all $x \in E$.*

PROPOSITION 1.2. *Let X be a Banach space. Then the following assertions are equivalent:*

(a) *X is a π space.*

(b) *There is a uniformly bounded net* $\{T_\alpha\}$ *of projections with finite dimensional ranges which tends strongly to the identity on* X.

(c) *There is a* $\lambda \geqq 1$ *satisfying the following property: for every finite dimensional subspace* $E \subset X$ *there is a projection* T *with finite dimensional range on* X *such that* $Tx = x$ *for all* $x \in E$ *and* $\| T \| \leqq \lambda$.

In the sequel we will use any one of the equivalent definitions of these properties without any reference. Let us now remark that in the case of separable Banach spaces, each of the last three properties (2), (3) and (4) implies the preceding one.

All the classical Banach spaces have the b.a.p. but it is unknown whether every Banach space shares this property. It is also an open problem whether any two of the above four properties are equivalent (in the separable case, say). The purpose of this paper is to investigate the connections among the structures of a Banach space X, its dual X^* and its second dual X^{**} (provided that at least one of those spaces has a structure). In Section 3 we investigate the effect of the structure of X^{**} on that of X. We prove, for example, that if X^{**} is a π_λ space then X is a $\pi_{\lambda+\varepsilon}$ space for every $\varepsilon > 0$. In Section 4 the relations between the structures of X and X^* are studied. We prove the following

THEOREM 1.3. *Let X be a separable Banach space. Then X has the f.d.d.p. if any of the following hold:*

(a) X *is a* π *space and* X^* *satisfies the b.a.p.*

(b) X^* *is a* π *space*

(c) X *is a* π *space and* X *is isomorphic to a conjugate Banach space*

(d) *There exists a sequence* $\{P_n\}$ *of finite rank projections on* X *such that* $P_mP_n = P_m$ *for all* m, n *with* $m \leqq n$ *and* $P_nx \to x$ *weakly for all* $x \in X$.

Moreover, if X^* *is separable and either* (a) *or* (b) *holds, then* X *has a shrinking finite dimensional decomposition.*

THEOREM 1.4. *Let X be a Banach space; then X has a shrinking basis (and consequently* X^* *has a boundedly complete basis) if either of the following holds:*

(a) X^* *has a basis*

(b) X *has a basis and* X^* *is separable and has the b.a.p.*

(See the definitions in Section 2 below.)

Thus in particular, we answer affirmatively the question raised by S. Karlin in [6]: if X^* has a basis, does X have a basis? We also partially solve J. R. Retherford's Problem 1 of [14]. Finally, in Section 5 we apply our results to show that every separable $\mathscr{L}_p$ space has a basis.

2. Definitions, notations and preliminary lemmas

In this section we explain the terminology we use and state several auxiliary lemmas. These lemmas are of a technical nature; they are easy and are demonstrated by known techniques. Some of the proofs will be therefore omitted.

In this paper an *operator* (resp. projection) always means a *bounded linear operator* (resp. projection). Two Banach spaces X and Y are called *isomorphic* (denoted by $X \approx Y$) if there is an invertible operator from X onto Y. The distance coefficient $d\{X, Y\}$ of two isomorphic Banach spaces is defined by $\inf(\| T\| \cdot \|T^{-1}\|)$ where the inf is taken over all invertible operators T from X onto Y. Let A be a set of elements of a Banach space X. We denote by span A the closed linear subspace spanned by A in X.

Let X be a Banach space and let E be a subspace of X. We denote by I_E the identity on E. Let T be an operator on X, then $T|_E$ denotes the restriction of T to E.

Let E be a subspace of a Banach space X. Then $E^\perp$ denotes the annihilator of E in $X^*(=\{x^* \in X^*: x^*(e) = 0 \text{ for all } e \in E\})$. If $F \subset X^*$ then $F_\perp$ denotes the annihilator of F in $X(=\{x \in X: f(x) = 0 \text{ for all } f \in F\})$. Let $\{x_n\}$ be a basis of a Banach space X. The sequence $\{f_n\}$ of functionals defined by $f_n(\sum_1^\infty a_i x_i) = a_n$ is called the *biorthogonal sequence* of the basis $\{x_n\}$. It is well known (see e.g. [1] p. 67) that $\{f_n\} \subset X^*$ and that Q_n, the natural projections of the basis, are uniformly bounded. We call $b = sup_n \| Q_n \|$ the *basis constant.* (Similarly, let X have a finite dimensional decomposition determined by the projections $\{Q_n\}$, then $\sup_n \| Q_n \|$ is finite and is called the decomposition's constant).

Given Banach spaces A and B, $A \oplus B$ denotes the Banach space $A \times B$ under the obvious operations with norm $\|(a, b)\| = \max (\| a \|, \| b \|)$. Given $1 \leqq p < \infty$ and Banach spaces $X_1, X_2, \cdots, (\sum \oplus X_n)_p$ denotes the Banach space consisting of all sequences (x_n) with $x_n \in X_n$ for all n and

$$\|(x_n)\| = (\sum \|x_n\|^p)^{1/p} < \infty .$$

LEMMA 2.1. (*a*) *Let X and Y be isomorphic Banach spaces with $d\{X, Y\} < c$ and let X have a basis with constant b. Then Y has a basis with constant $< bc$.*

(b) *Let X be a Banach space with a basis $\{x_n\}$. Assume that the basis' constant is b. Then the biorthogonal functionals form a basis with constant $\leqq b$ in the subspace spanned by them in X^*.*

(c) *Let E be a finite dimensional subspace of a Banach space X and let P be a projection from X onto E with $\| P \| = M$. Assume that E has a basis $\{e_n\}$ with constant b. Then $P^*(X^*)$ has a basis with a constant $\leqq Mb$.*

LEMMA 2.2. *Let a Banach space X have a finite dimensional decomposition determined by the projections $\{Q_n\}$. Let each subspace $(Q_n - Q_{n-1})(X)$ have a basis $\{x_i^n\}_{i=1}^{d(n)}$ with constant b_n such that $\sup_n b_n = b < \infty$. ($d(n) = \dim(Q_n - Q_{n-1})(X)$ and Q_0 is the zero operator.) Then the sequence $x_1^1, x_2^1, \cdots x_{d(1)}^1, x_1^2, x_2^2, \cdots x_{d(2)}^2, x_1^3, \cdots$ forms a basis for X.*

Let $\{F_n\}$ be a finite dimensional decomposition of a Banach space X and let $\{Q_n\}$ be its natural projections. The decomposition is called *shrinking* if $X^* = \operatorname{span}\{Q_n^*(X^*)\}_{n=1}^{\infty}$. A basis $\{x_n\}$ of X is called *shrinking* if its biorthogonal functionals $\{f_n\}$ span X^*. The basis $\{x_n\}$ is called *boundedly complete* if $\sum_1^\infty a_i x_i$ converges whenever $\sup_n \| \sum_{i=1}^n a_i x_i \| < \infty$.

It is known (see e.g. [1] p. 70–71) that $\{x_n\}$ is a shrinking basis if and only if $\{f_n\}$ is a boundedly complete basis for X^*.

Let X be a Banach space, let E_1 and E_2 be subspaces of X and let $\varepsilon > 0$. We say that E_2 is *ε-close* to E_1 if there is an invertible operator T from E_1 onto E_2 with $\| Tx - x \| \leqq \varepsilon \| x \|$ for all $x \in E_1$.

LEMMA 2.3. *Let X be a Banach space and let E_1, E_2, F_1 and F_2 be closed subspaces of X. Assume that for each $i = 1,2$ $F_i \subset E_i$ and there is a projection B_i from E_i onto F_i with $\| B_i \| = c_i$. Let δ and ε be positive numbers such that $\delta c_1 + \varepsilon(1 + c_1) \leqq \frac{1}{2}$ and let F_2 be δ-close to F_1 and E_2 ε-close to E_1. Then $d\{(I - B_1)(E_1), (I - B_2)(E_2)\} \leqq 3(1 + c_1)(1 + c_2)$.*

PROOF. Let $U: E_1 \to E_2$ and $V: F_1 \to F_2$ be surjective one-to-one operators such that $\| I_{E_1} - U \| \leqq \varepsilon$ and $\| I_{F_1} - V \| \leqq \delta$. Define $T: E_1 \to E_2$ by $Tx = VB_1 x + U(x - B_1 x)$ for all $x \in E_1$. Fixing $x \in E_1$ we have that

$$\|Tx - x\| \leqq \|VB_1 x - B_1 x\| + \|U(x - B_1 x) - (x - B_1 x)\| \leqq \delta c_1 \|x\| + \varepsilon(1 + c_1)\|x\| \leqq (\tfrac{1}{2})\|x\|.$$

It follows that T is an invertible operator carrying E_1 onto E_2 and F_1 onto F_2 such that $\| T \| \| T^{-1} \| \leqq 3$, hence $d(E_1/F_1, E_2/F_2) \leqq 3)$. Moreover, for $i = 1,2$ $d((I - B_i)(E_i), E_i/F_i) \leqq \| I - B_i \| \leqq 1 + c_i$ since if ϕ_i is the natural isomorphism from E_i/F_i onto $(I - B_i)(E_i)$, then $\| \phi_i \| = \| I - B_i \|$ and $\| \phi_i^{-1} \| = 1$. Thus $d((I - B_1)(E_1), (I - B_2)(E_2)) \leqq d((I - B_1)(E_1), E_1/F_1) \cdot d(E_1/F_1, E_2/F_2)$. $d(E_2/F_2, (I - B_2)(E_2)) \leqq 3(1 + c_1)(1 + c_2)$.

LEMMA 2.4. *Let T be an operator from a Banach space X onto an n-dimensional subspace $E \subset X$. Let $k \leqq n$ and let F be a k-dimensional subspace of X such that $\| T|_F - I_F \| < \varepsilon < 1$, where $(1-\varepsilon)^{-1}\varepsilon k < 1$. Then*

(a) *there is an operator S from X onto an n-dimensional subspace of X such that $S|_F = I_F$, $\| S - T \| < (1-\varepsilon)^{-1}\varepsilon k \| T \|$, and $S^*(X^*) = T^*(X^*)$.*

(b) *If, in addition, T is a projection then S can be chosen to be a projection and $\| S|_{T(X)} - I_{T(X)} \| < (1-\varepsilon)^{-1}\varepsilon k$.*

PROOF. Obviously the restriction $U = T|_F$ is an invertible operator from F onto $T(F)$ with $\| U \| < 1 + \varepsilon$ and $\| U^{-1} \| < (1-\varepsilon)^{-1}$; hence $\| U^{-1} - I_{T(F)} \| < \varepsilon(1-\varepsilon)^{-1}$. It is known that there always exists a projection from a Banach space onto its k-dimensional subspace, of norm less than or equal to k (e.g., this follows from [15]). Let P be a projection from $E = TX$ onto TF with $\| P \| \leqq k$, put $V = U^{-1}P + I_E - P$, and put $S = VT$. We have that $\| V - I_E \| = \| U^{-1}P - P \| = \| (U^{-1} - IT(F)P \| < k\varepsilon(1-\varepsilon)^{-1} < 1$, hence $V: E \to X$ is a one-to-one operator, thus T and S have the same null space which implies that T^* and S^* have the same range (in view of the finite codimensionality of the null space of T). We have that $S|_F = I_F$ and S is a projection if T is. The remaining assertions of the lemma follow easily from the definition of S.

3. Local reflexivity of Banach spaces and its consequences

In this section we will be interested in the connections between the structure of X^{**} and that of X. It has been recently proved ([10] Theorem 3.1) that the finite dimensional subspaces of X^{**} are "almost the same" as those of X. Let us state this result in a slightly stronger version (which is still yielded by the argument of [10]); namely:

The principle of local reflexivity: Let X be a Banach space (regarded as a subspace of X^{**}) let U and F be finite dimensional subspaces of X^{**} and X^*, respectively, and let $\varepsilon > 0$. Then there exists a one-to-one operator $T: U \to X$ with $T(x) = x$ for all $x \in X \cap U$, $f(Te) = e(f)$ for all $e \in U$ and $f \in F$ and $\| T \| \, \| T^{-1} \| < 1 + \varepsilon$.

A MODIFICATION OF THE PROOF OF [10]: Given a subset L of a Banach space Y, let $\tilde{L}$ denote the weak* closure of L and L the norm-interior of $\tilde{L}$ in Y^{**}. Observe that if the space Y and its subsets $U_1, U_2, \cdots, U_n, L$, and $\{y_0\}$ are given where $U_1, \cdots, U_n$ are open convex, $L + y_0$ is a closed linear subspace of finite codimension and $\tilde{L} \cap U_1 \cap \cdots \cap U_n \neq \phi$, then $L \cap U_1 \cap \cdots \cap U_n \neq \phi$. Indeed we may assume $y_0 = 0$; then for any open convex subset V of Y $V \cap \tilde{L} \subset \widetilde{V \cap L}$ (see Appendix). Hence $V \cap \tilde{L}$ is contained in the norm-interior of $\widetilde{V \cap L}$ relative to $\tilde{L}$. This enables us to assume w.l.g. that $L = Y$. Should $U_1 \cap \cdots \cap U_n = \phi$, by a theorem of Klee (c.f. [10], p. 348) there would exist a linear map S from Y onto a finite dimensional space B such that $\bigcap_{i=1}^n S(U_i) = \phi$. Whence since $S^{**}(U_i) = S(U_i)$ ([10], p. 332), $\bigcap_{i=1}^n U_i = \phi$, a contradiction. Now let $k, u_1, \cdots, u_k$ and $C_1, \cdots, C_m, K_1, \cdots K_m$ be defined as in [10], p. 333 and put $L = \{(x_1, \cdots, x_k): f(x_i) = u_i(f)$ for all $f \in F$ and all $1 \leqq i \leqq k\}$. The existence of T follows if $L \cap \bigcap_{i=1}^m C_i \cap K_i \neq \phi$. Obviously there is a $y \in X^k$ so that $L + y$ is a closed linear subspace of finite codimension in X^k. But $(u_1, \cdots, u_k) \in \tilde{L} \cap \bigcap_{i=1}^m K_i^{**} \cap C_i^{**}$, where K_i^{**} and C_i^{**} are defined as in [10], $K_i^{**} \subset \boldsymbol{K}_i$ and $C_i^{**} \subset \boldsymbol{C}_i$. Thus $\tilde{L} \cap \bigcap_{i=1}^m \boldsymbol{K}_i \cap C_i \neq \phi$, hence $L \cap \bigcap_{i=1}^m K_i \cap C_i \neq \phi$ by our initial observation.

The next result is essentially proved in ([5] Lemma 1) by using tensor product methods. Our proof is based on the principle of local reflexivity.

LEMMA 3.1. *Let X and Y be Banach spaces with* $\dim Y < \infty$. *Let F be a finite dimensional subspace of X^*, let R be an operator from X^* into Y and let $\varepsilon > 0$. Then there is a weak * continuous operator S from X^* to Y such that*

(a) $S|_F = R|_F$

(b) $\|S\| \leqq \|R\|(1 + \varepsilon)$

(c) $S^* y^* = R^* y^*$ *whenever $R^* y^*$ belongs to X (X being regarded as a subspace of X^{**}).*

PROOF. The principle of local reflexivity yields the existence of an invertible operator T from $E = \operatorname{Range} R^*$ into X such that $T|_{E \cap X} = I_{E \cap X}$, $f(Te) = ef$ for all $e \in E$ and $f \in F$ and $\|T\|, \|T^{-1}\| < 1 + \varepsilon$. The operator $S = [TR^*]^*$ maps X^* into $Y^{**} = Y$, it is obviously weak * continuous and if $f \in F$, $y^* \in Y^*$ then $Sf(y^*) = f(TR^* y^*) = (R^* y^*)(f) = y^*(Rf)$ and hence $Sf = Rf$. Also, $\|S\| = \|TR^*\| \leqq \|T\| \|R\| \leqq (1 + \varepsilon)\|R\|$. Finally, if $y^* \in Y^*$ and $R^* y^* \in X$ then $R^* y^* = TR^* y^* = S^* y^*$. This completes the proof.

COROLLARY 3.2. *If, in addition to the assumptions of the preceding lemma we have that $Y \subset X^*$ then there is an operator T on X with $T^* = S$, where S is the operator constructed in Lemma 3.1. If, moreover, R is a projection then S (and hence, also T) can be chosen to be a projection. In both cases $S^* x^{**} = R^* x^{**}$ whenever $R^* x^{**} \in X$.*

PROOF. The first assertion is an easy and well known consequence of the weak * continuity of S. To prove the second assertion, let R be a projection, replace F by $F' = \operatorname{span}\{F \cup Y\}$ and construct S as before. Then $S|_{F'} = R|_{F'}$ and hence $Sy = Ry = y$ for all $y \in Y$. The last assertion follows from (c) of Lemma 3.1.

We now strengthen the principle of local reflexivity as follows:

THEOREM 3.3. *Let X be a Banach space (regarded as a subspace of X^{**}), let E and G be finite dimensional subspaces of X^{**} and X^*, respectively, and let $1 > \varepsilon > 0$. Assume that there is a projection P from X^{**} onto E with $\|P\| = M$. Then there is a one-to-one operator T from E into X and a projection P_0 from X onto $T(E)$ such that*

(a) *$Te = e$ for all $e \in E \cap X$*

(b) *$f(Te) = e(f)$ for all $e \in E$ and $f \in G$.*

(c) $\|T\| \, \|T^{-1}\| \leqq 1 + \varepsilon$

(d) $\|P_0\| \leqq M(1 + \varepsilon)$.

*If, in addition, P is weak * continuous then P_0 can be chosen to satisfy* (a), (b), (c), (d) and

(e) *$P_0^{**}x^{**} = Px^{**}$ for all $x^{**} \in X^{**}$ for which $Px^{**} \in X$.*

PROOF. By Corollary 3.2 there is a projection Q on X^* such that $Q^*(X^{**}) = E$ and $\|Q\| \leqq (1 + \frac{1}{4}\varepsilon)\|P\|$. In view of the principle of local reflexivity, putting $F = Q(X^*)$, there is an operator T from E into X such that $f(Te) = e(f)$ for all $e \in E$ and $f \in \operatorname{span}\{F \cup G\}$, $T|_{E \cap X} = I_{E \cap X}$, and $\|T\|, \|T^{-1}\| < 1 + \frac{1}{4}\varepsilon$.

Now put $P_0 = TQ^*|_X$. Then $\|P_0\| \leqq \|T\| \, \|Q^*\| = \|T\| \, \|Q\| \leqq (1+\varepsilon)\|P\|$. If $e \in E$ and $x^* \in X^*$, then $(Q^*Te)x^* = (Qx^*)(Te) = e(Qx^*) = (Q^*e)x^* = e(x^*)$; whence $Q^*Te = e$, and hence $TQ^*(Te) = Te$. Hence $P_0|_{T(E)} = I_{T(E)}$ and of course $P_0(X) \subset T(E)$, so P_0 is the desired projection onto $T(E)$.

To prove (e), let P be weak * continuous; then we may assume that $Q^* = P$. By Lemma 3.1 (c) and Corollary 3.2 the projection P_0 constructed above satisfies the equality $P_0^{**}x^{**} = Q^*x^{**}$ whenever $Q^*x^{**} \in X$. This proves Theorem 3.3.

COROLLARY 3.4. *Let X^{**} be a π_λ space. Then X is a $\pi_{\lambda+\varepsilon}$ space for every $\varepsilon > 0$.*

PROOF. Let E_0 be a finite dimensional subspace of X (X regarded as a subspace of X^{**}). Since X^{**} is a π_λ space, by Lemma 2.5, given $\varepsilon > 0$, there is a finite dimensional subspace $E \subset X^{**}$ such that $E_0 \subset E$ and a projection P from

X^{**} onto E with $\| P \| \leqq \lambda + \varepsilon/2$. In view of Theorem 3.3 there is a one-to-one operator $T: E \to X$ and a projection Q on X such that $T|_{E_0} = I_{E_0}$, $Q(X) = T(E)$ and $\| Q \| \leqq \lambda + \varepsilon$. It follows that X is a $\pi_{\lambda+\varepsilon}$ space.

4. The structure of X and X^*

In this section we investigate the relations between the structure of X and that of X^*. We begin with

THEOREM 4.1. *Let X be a separable Banach space. Then the following assertions are equivalent*:

(a) *X has a f.d.d.*

(b) *There are a separable subspace $Y \subset X^*$ and sequences $\{T_n\}$ and $\{P_n\}$ of operators with finite dimensional ranges satisfying the following conditions*:

(4.1) $P_n: X \to X$, $P_n^*(X^*) \subset Y$ *and* $T_n: X^* \to Y$ *for all* n.

(4.2) $P_n x \to x$ *for all* $x \in X$, $T_n y \to y$ *for all* $y \in Y$ *and* $\sup_n \| T_n \| < \infty$.

(4.3) *The operators $\{P_n\}$ are projections*

(c) *Exactly the same as* (b), *except that* (4.3) *is replaced by the condition*

(4.4) *The operators $\{T_n\}$ are projections.*

Moreover, if either (b) *or* (c) *hold, the sequence $\{Q_i\}$ determining the f.d.d. for X may be chosen such that $\{Q_i^* |_Y\}$ determines a f.d.d. for Y.*

PROOF. It is obvious that (a) implies (b) and (c). Let us prove the implication (b) $\to$ (a). It suffices to show the existence of a uniformly bounded sequence $\{Q_n\}$ of projections on X with finite dimensional ranges spanning X such that $Q_nQ_k = Q_kQ_n = Q_{\min(k,n)}$ (indeed then, the subspaces $F_1 = Q_1(X)$, $F_n = (Q_n - Q_{n-1}(X)$ for $n > 1$ form a finite dimensional decomposition for X). The main tool in the proof is the following

LEMMA 4.2. *Let X be a Banach space satisfying* (b) *above, let E and F be finite dimensional subspaces of X and Y respectively and let $1 > \varepsilon > 0$. Then there is a projection Q with finite dimensional range on X such that*

(4.5) $Qe = e$ *for all* $e \in E$

(4.6) $Q^*f = f$ *for* $f \in F$.

(4.7) $Q^*(X^*) \subset Y$

(4.8) $\|Q\| \leqq 2c + 2K + 4cK$ *where* $c = \sup_n \|P_n\|$ *and* $K = \sup_n \|T_n\|$

(4.9) $Q(X)$ *is* ε-*close to* $P_{n(i)}(X)$ *for some integer* $n(i)$.

We will prove Lemma 4.2 after we complete the proof of Theorem 4.1 to which we now return. Let $\{x_n\}$ and $\{y_n\}$ be dense sequences in X and Y respectively with $x_1 = y_1 = 0$ and let $\{\varepsilon_n\}$ be a sequence of positive numbers. We proceed by induction: let $Q_1 = P_1$, let $k \geqq 1$, and suppose that the projections $Q_1, Q_2, \cdots, Q_k$ on X have been chosen such that the following conditions are satisfied for all $1 \leqq i, j \leqq k$

(4.10) $Q_iQ_j = Q_jQ_i = Q_{\min(i,j)}$

(4.11) $Q_i(X) \supset \{x_1, x_2, \cdots, x_i\}$

(4.12) $Y \supset Q_i^*(X^*) \supset \{y_1, y_2, \cdots, y_i\}$

(4.13) $\|Q_i\| \leqq 2c + 2K + 4cK$ where $c = \sup_n \|P_n\|$ and $K = \sup_n \|T_n\|$

(4.14) $Q_i(X)$ is ε_i-close to $P_{n(i)}(X)$ for some integer $n(i)$.

Now let E be the finite dimensional subspace spanned by $\{x_{k+1}\} \cup Q_k(X)$ in X and let $F = \text{span}(\{y_{k+1}\} \cup Q_k^*(X^*))$ in Y. By Lemma 4.2 there is a projection $Q = Q_{k+1}$ with finite dimensional range on X such that (4.5)-(4.9) are satisfied with $\varepsilon = \varepsilon_{k+1}$ and some integer $n = n(k+1)$. It follows that (4.13) and (4.14) are satisfied for $i = k+1$ while (4.11) and (4.12) are yielded by (4.5), (4.6) and the induction hypothesis. If $i \leqq k$ then $Q_{k+1}Q_i = Q_i$ and $Q_{k+1}^*Q_i^* = Q_i^*$ and hence $Q_iQ_{k+1} = Q_{k+1}Q_i = Q_i$. The sequence $\{Q_i\}_{i=1}^{\infty}$ obviously satisfies (4.10)–(4.14) and therefore it defines a finite dimensional decomposition for X while $\{Q_k^*(X^*)\}_{k=1}^{\infty}$ span Y and hence the restrictions $Q_k^*|_Y$ determine a decomposition for Y. This proves the implication $(b) \to$ (a). The proof of the implication (c) $\to$ (a) is similar and is based on the following analogue of Lemma 4.2:

LEMMA 4.3. *Let X be a Banach space satisfying* (c), *let E and F be finite dimensional subspaces of X and Y respectively and let $1 > \varepsilon > 0$. Then there exists a projection Q with finite dimensional range on X such that*

(4.15) $Qe = e$ *for all* $e \in E$

(4.16) $Qf = f$ *for all* $f \in F$

(4.17) $Q^*(X^*) \subset Y$

(4.18) $\|Q\| \leqq 2c + 2K + 4cK$ where $c = \sup_n \|P_n\|$ *and* $K = \sup_n \|T_n\|$

(4.19) $Q^*(X^*)$ *is* ε-*close to* $T_n(X^*)$ *for some integer* n.

The proof of the lemma will be given later. As in the first part, assume that the projections $Q_1, Q_2, \cdots, Q_k$ on X have been chosen such that (4.10)–(4.13) are satisfied in addition to the condition

(4.20) $\qquad Q_i^*(X^*)$ is ε_i-close to $T_{n(i)}(X^*)$ for some $n(i)$. $1 \leqq i \leqq k$.

Put $E = \mathrm{span}(\{x_{k+1}\} \cup Q_k(X))$ and $F = (\{y_{k+1}\} \cup Q_k^*(X^*))$; then by using Lemma 4.3 (in the same way Lemma 4.2 was used in the first part) one can find a projection Q_{k+1} satisfying (4.10)–(4.13) and (4.20) with $i = k+1$. The sequence $\{Q_k\}$ obviously determines a f.d.d. for X while $\mathrm{span}\{Q_k^*(X^*)\}_{k=1}^{\infty} = Y$ and hence the restrictions $Q_k^*|_Y$ determine a f.d.d. for Y. This proves Theorem 4.1.

Proof of Lemma 4.2. Let δ be a positive number such that $\delta \dim(F) < \frac{1}{4}$ and let m be so large that $\| T_m f - f \| \leqq \delta \| f \|$ for all $f \in F$. The proof of Lemma 2.4 then shows that there is an operator T with finite dimensional range from X^* to Y such that $Tf = f$ for all $f \in F$ and $\| T \| \leqq (3/2)K$. By Lemma 3.1 there is an operator S on X satisfying the equalities $S^*(X^*) = T(X^*)$ and $S^*f = Tf = f$ for all $f \in F$ and having norm $\| S \| \leqq 2K$; then of course S has finite dimensional range since S^* does. Now let G denote the span of E and range S in X. Then G is finite dimensional; let γ be a positive number smaller than $\frac{1}{2}$ satisfying $\gamma \dim G < \varepsilon/4$ and choose n so large that $\| P_n g - g \| = \leqq \gamma \| g \|$ for all $g \in G$. Then by Lemma 2.4 (b) there is a projection P with finite dimensional range on X such that $Pg = g$ for all $g \in G$, $\| P \| \leqq 3/2 \|P_n\| \leqq \frac{3}{2}c$, $P(X)$ is ε-close to $P_n(X)$ and P^* and P_n^* have the same range. We claim that $Q = S + P - SP$ is a projection on X having all the desired properties. Evidently (4.7) and (4.8) hold immediately, and of course Q has finite dimensional range.

We have that $I - Q = (I-S)(I-P)$, whence $I - Q^* = (I-P^*)(I-S^*)$. Since range $S \subset \mathrm{range}\, P$, null $S^* = (\mathrm{range}\, S)^{\perp} \supset \mathrm{null}\, P^* = (\mathrm{range}\, P)^{\perp}$, whence $(I-S^*)\big|_{null\, P^*} = I\big|_{null\, P^*}$, and hence $(I-P^*)(I-S^*)$ is a projection onto the null space of P^*. Hence Q^* and thus Q is indeed a projection; moreover range Q = (null $Q^*)_{\perp}$ = (null $P^*)_{\perp}$ = range P, whence (4.9) holds by the definition of P. $I - Q^* = 0$ on F since $I - S^* = 0$ on F, hence (4.6) holds, and $I - Q = 0$ on G, thus (4.5) holds (since $G \supset E$), so the proof of Lemma 4.2 is complete.

Proof. of Lemma 4.3. The arguments here are very similar to those of the preceding proof. Let δ satisfy $0 < \delta \cdot \dim(E) < \frac{1}{2}$ and choose m so large that $\| P_m e - e \| \leqq \delta \| e \|$ for all $e \in E$. By Lemma 2.4 there is an operator P with finite dimensional range on X such that $Pe = e$ for all $e \in E$, $P^*(X^*) = P_m^*(X^*) \subset Y$

and $\| P \| \leqq 2c$. Let H denote the span of $P^*(X^*)$ and F in Y, let γ satisfy the inequalities $0 < \gamma \cdot \dim(H) < \varepsilon/4$ and choose n large enough so that $\| T_n h - h \| = \leqq \gamma \| h \|$ for all $h \in H$. By Lemma 2.4(b) there is a projection T on X^* such that $Th = h$ for all $h \in H$, $T(X^*) \subset Y$, $\| T \| \leqq \frac{3}{2}K$ and such that $T(X^*)$ is ε-close to $T_n(X^*)$. Using Lemma 3.1 one can find a projection S on X such that $S^*(X^*) = T(X^*)$, $S^*|_H = I_H$ and $\| S \| \leqq 2K$. As in Lemma 4.2 it can be shown that $(I-S)(I-P)$ is a projection from X onto $(I-S)(X)$ and that

$$Q = I - (I-S)(I-P) = S + P - SP$$

is the desired projection. Lemma 4.3 is thus proved.

The proofs of Lemma 4.2 and Lemma 4.3 yield, respectively, the following two results:

LEMMA 4.4. *Let X be a π_λ space and X^* have the μ-m.a.p. Assume that E and F are finite dimensional subspaces of X and X^* respectively and let $\varepsilon > 0$. Then there is a projection Q with finite dimensional range on X such that $Q|_E = I_E$, $Q^*|_F = I_F$, $\| Q \| \leqq 2\mu + 2\lambda + 4\mu\lambda$ and such that $Q(X)$ is ε-close to some subspace E_α belonging to the family $\{E_\alpha\}$ which defines the π-structure of X.*

LEMMA 4.5. *Let X be a Banach space and assume that X^* is a π_λ space. Let E and F be finite dimensional subspaces of X and X^* respectively and let $\varepsilon > 0$. Then there is a projection Q with finite dimensional range on X such that $Q|_E = I_E$, $Q|_F = I_F$, $\| Q \| \leqq 4\lambda + 4\lambda^2$ and such that $Q^*(X^*)$ is ε-close to one of the subspaces E_α which define the π-structure of X^*.*

The proof of Lemma 4.4 is essentially the same as that of Lemma 4.2. To prove Lemma 4.5 we will show that X has the λ-m.a.p. Once this is accomplished, the lemma will follow from the proof of Lemma 4.3. By the definition of the π_λ spaces there is a net $\{T_\alpha\}_{\alpha \in A}$ of projections with finite dimensional ranges on X such that $T_\alpha \to I$ strongly and $\| T_\alpha \| \leqq \lambda$ for every $\alpha \in A$. Let $\lambda' > \lambda$, then, for each α there exists, by Lemma 3.1, a projection S_α on X such that $S^*_\alpha(X^*) = T_\alpha(X^*)$ and $\| S_\alpha \| \leqq \lambda'$. It follows that $S^*_\alpha \to I$ strongly and hence $S_\alpha x \to x$ weakly for every $x \in X$. Standard arguments (see e.g. [2] p. 477) show that there is a net $\{P_\beta\}$ of operators on X such that $P_\beta \to I$ strongly and each P_β is of the form $P_\beta = \sum_{i \in \sigma(\beta)} a_i S_i$, where $\sigma(\beta)$ is a finite subset of A, $a_i \geqq 0$ and $\sum a_i = 1$. Obviously $\| P_\beta \| \leqq \lambda'$ and hence X has the λ'-m.a.p. for every $\lambda' > \lambda$, which implies that X has the λ-m.a.p.

REMARK 4.6. The sequence $\{\varepsilon_k\}$ appearing in the proof of Theorem 4.1 as well as conditions (4.9), (4.14) and (4.19) are of no importance there. However, they play a significant role in the following sense: Our proof shows that the natural projections of the f.d.d. we construct can be chosen so that the subspaces $Q_k(X)$ (resp. $Q_k^*(X^*)$) are as close as we wish to "nice" finite dimensional subspaces of the π-structure of X (resp. the structure of Y).

Let us now pass to the

PROOF OF THEOREM 1.3. Let (a) be satisfied, then, by Lemma 4.4 a sequence $\{Q_k\}$ of projections with finite dimensional ranges on X can be constructed by induction, as in the proof of Theorem 4.1, such that $Q_nQ_k = Q_kQ_n = Q_{min(k,n)}$, $\operatorname{span}\{Q_k(X)\}_{k=1}^\infty = X$ and $\sup_n \| Q_n \| < \infty$. Obviously $\{Q_k\}$ defines a f.d.d. in X. Let (b) hold; then by Lemma 4.5 a sequence $\{Q_k\}$ with the same properties can be constructed. In both cases, if X^* happens to be separable, then the $\{Q_k\}$ can be chosen so that $\{Q_k^*\}_{k=1}^\infty$ defines a f.d.d. in X^*, i.e. X has a shrinking f.d.d. Hence, if (c) is satisfied and $X \approx Y^*$ for some Banach space Y then the preceding argument yields a f.d.d. in Y with natural projections $\{Q_k\}$ for which $\{Q_k^*\}$ defines a f.d.d. in Y^*. Since $X \approx Y^*$, X has the f.d.d.p. Finally, let (d) hold, let $\{d_n\}$ be a dense sequence in X and fix n. Since $(P_m d_1, \cdots P_m d_n)_n \to (d_1, \cdots, d_n)$ weakly in $X \oplus \cdots \oplus X$, there exist a positive integer $b(n)$ and non-negative scalars $a_i^n, 1 \leqq i \leqq b(n)$ with $\sum_{i=1}^{b(n)} a_i^n = 1$ and $a_{b(n)}^n > 0$, such that $\| \sum_{i=1}^{b(n)} a_i^n P_i d_j - d_j \| < 1/n$ for all $j, 1 \leqq j \leqq n$ (see e.g. [1] p. 40, Th. 2). Thus putting $T_n = \sum_{i=1}^{b(n)} a_i^n P_i$, $T_n \to I$ strongly. Moreover $T_n^*(X^*) = P_{b(n)}^*(X^*)$. For if $\sum_{i=1}^{b(n)} a_i^n P_i x = 0$ then $\sum_{i=1}^{b(n)} a_i^n P_1 P_i(x) = \sum_{i=1}^{b(n)} a_i^n P_1 x = P_1 x = 0$. Proceeding by induction we obtain that $(\sum_{i=1}^{b(n)} a_i^n) P_j x = 0$, whence $P_j x = 0$ for all $1 \leqq j \leqq b(n)$, hence T_n and $P_{b(n)}$ have the same null space. We also have immediately that $P_n^*(X^*) \subset P_{n+1}^*(X^*)$. Put $Y = \operatorname{span}\{P_n^*(X^*)\}_{n=1}^\infty$ then $P_n y \to y$ for all $y \in Y$ and $T_n^*(X^*) \subset Y$ for all n. Hence, by Theorem 4.1 X has the f.d.d.

REMARK 4.7. Theorem 1.3(d) is proved in [4] under the stronger assumption that $P_n \to I$ strongly. Let us also note that from Theorem 1.3 it follows in particular that every separable reflexive π space has a f.d.d.

Lemmas 4.4 and 4.5 yield the following

COROLLARY 4.8. *If X^* is a π space, so is X. If X is a π space and X^* has the b.a.p., then X^* is a π space.*

The above results deal with finite dimensional decompositions and weaker structures. Let us now discuss bases.

THEOREM 4.9. *Let X be a separable Banach space. Then the following assertions are equivalent:*

(a) *X has a basis*

(*b*) *There is a separable subspace $Y \subset X^*$ with a basis and sequences $\{S_n\}$ and $\{P_n\}$ of operators with finite dimensional ranges satisfying the following conditions:*

(i) $P_n: X \to X$, $P_n^*(X^*) \subset Y$ *and* $S_n: X^* \to Y$ *for all* n.

(ii) $P_n x \to x$ *for all* $x \in X$, $S_n y \to y$ *for all* $y \in Y$ *and* $\sup_n \| S_n \| < \infty$.

PROOF. The implication (a) $\to$ (b) is immediate. To prove (b) $\to$ (a), let $\{y_n\}$ be a basis of Y and let $\{U_n\}$ denote its natural projections ($U_n(\sum_1^\infty a_i y_i) = \sum_1^n a_i y_i$). Put $Y_n = \mathrm{span}\{y_1, y_2, \cdots, y_n\}$, fix n and choose k so large that $\| S_k y - y \| \leqq \delta \| y \|$ for all $y \in Y_n$, where δ is a positive number satisfying $\delta n < \frac{1}{2}$. Put $\gamma = \sup_n \| S_n \|$; then by Lemma 2.4, there is an operator S_n' from X^* to Y such that $S_n' y = y$ for all $y \in Y_n$ and $\| S_n' \| \leqq 2\gamma$. Let $b = \sup_m \| U_m \|$; then the operator $T_n = U_n S_n'$ is a projection from X^* onto Y_n with $\| T_n \| \leqq 2b\gamma = c$. It is easy to see that $T_n y \to y$ for all $y \in Y$ and hence condition (c) of Theorem 4.1 is satisfied. By Theorem 4.1 there is a f.d.d. of X determined by a sequence $\{Q_k\}$ of commuting projections such that $\sup_k \| Q_k \| \leqq 2c + 2K + 4cK$, where $K = \sup_n \| P_n \|$. Moreover, given $\varepsilon > 0$, the projections Q_k can be chosen (see (4.20)) so that for each k, $Q_k^*(X^*)$ is ε-close to a subspace $T_{n(k)}(X^*) = Y_{n(k)}$. In view of Lemma 2.2, in order to complete the proof it suffices to show that each of the decomposition's subspaces $(Q_k - Q_{k-1})(X)$ has a basis with constant b_k such that $\sup_k b_k < \infty$. To do this put $M = 2c + 2K + 4cK$, assume that $\varepsilon(1 + M) < \frac{1}{4}$ and fix k. Consider the spaces $E_1 = Y_{n(k)}$, $E_2 = Q_k^*(X^*)$, $F_1 = Y_{n(k-1)}$ and $F_2 = Q_{k-1}^*(X^*)$. Put $B_1 = U_{n(k-1)} |_{Y_{n(k)}}$ and $B_2 = Q_{k-1}^* |_{E_2}$; then for each $i = 1, 2$, B_i is a projection from E_i onto F_i, $\| B_1 \| \leqq c$, $\| B_2 \| \leqq 2c + 2K + 4cK = M$ and $\varepsilon \| B_1 \| + \varepsilon(1 + \| B_2 \|) \leqq 2\varepsilon(1 + M) < \frac{1}{2}$. Since E_2 and F_2 are ε-close to E_1 and F_1 respectively, the assumptions of Lemma 2.3 are satisfied. Therefore there is an invertible operator V from $(I - B_1)(E_1) = \mathrm{span}\{y_i\}_{i=n(k-1)+1}^{n(k)}$ onto $(I - B_2)(E_2) = (Q_k^* - Q_{k-1}^*)(X^*)$, such that $\| V \| \, \| V^{-1} \| \leqq 3(1 + \| B_1 \|)(1 + \| B_2 \|) \leqq 3(1 + c)(1 + M) = M_0$. But $(I - B_1)(E_1)$ has a basis $y_{n(k-1)+1}, y_{n(k-1)+2}, \cdots, y_{n(k)}$ with constant $\leqq b$ and hence, by Lemma 2.1(a) $(Q_k^* - Q_{k-1}^*)(X^*)$ has a basis with constant $\leqq M_0 b$. It follows from Lemma 2.1(c) that each decomposition subspace $(Q_k - Q_{k-1})(X)$ has a basis with constant $\leqq 2bMM_0$. This concludes the proof of Theorem 4.9.

REMARK 4.10. It follows from the above proof that $\{Q_k^*[X^*]\}$ spans Y, and hence the biorthogonal functionals to the basis constructed for X, form a basis for Y. It then follows easily (cf. the proof of Theorem 1.4(b) below) that if X has a basis and X^* satisfies the b.a.p., then for every separable $Y \subset X^*$ there is a basis (y_n) of X with biorthogonal functionals (y_n^*) such that span$\{y_n^*\}_{n=1} \supset Y$.

From Theorem 4.9 easily follows the

PROOF OF THEOREM 1.4. (a) Let X^* have a basis $\{y_n\}$ and let T_n denote its natural projections. Let $\sup_n \| T_n \| = K$; then by Lemma 3.1, for each n there is a projection S_n on X such that $S_n^*(X^*) = T_n(X^*)$ and $\| S_n \| \leqq 2K$. Obviously $S_n^* \to I$ strongly and hence $S_n x \to x$ weakly for every $x \in X$. As was mentioned before, there are convex combinations $P_n = \sum_{i \in \sigma(n)} a_i S_i$ which tend strongly to I on X. Put $Y = X^*$; then, as remarked before, our proof of Theorem 4.9 shows that X has a basis $\{x_n\}$ such that its biorthogonal functionals are a basis of Y.

(b) Let X^* be separable and have the b.a.p. and let U_n be the natural proections of the given basis of X. Put $Z = X^*$, $Y = X, X$ regarded as a subspace of $X^{**} = Z^*$ and $T_n = U_n^{**}$. Using Lemma 3.1 it is easy to show that assumption (b) of Theorem 4.9 is satisfied with Z equal to the "X" of Theorem 4.9, and hence Z has a basis. By part (a) of this theorem, since $Z = X^*$, we get that X has a shrinking basis. Theorem 1.4 is thus proved.

REMARK 4.11. A Banach space B is said to have the approximation property if for every compact set $K \subset B$, and $\varepsilon > 0$, there exists an operator T on B with finite dimensional range such that $\| Tk - k \| < \varepsilon$ for all $k \in K$. It follows from the results of [3] (see Th. 8 §4 and Props. 35, B2 and 40 §5) that if X^* is separable and has the approximation property, then X^* has the 1-m.a.p. (and hence, in particular, X^* has the b.a.p.). Thus by Theorem 1.4, if X has a basis and X^* is separable and satisfies the approximation property, then X^* has a boundedly complete basis; on the other hand if there exists a Banach space failing the approximation property then by a recent result of Lindenstrauss, there exists a space X with a basis such that X^* is separable and fails the approximation property (cf. [8], Corollary 3 and the remark preceding Corollary 4). Thus the question: "For all Banach spaces X, if X has a basis and X^* is separable, does

X^* have a basis?" is equivalent to the question: "Does every Banach space satisfy the approximation property?"

It was observed by Lindenstrauss in [8] that our Theorem 1.4 and his results together, imply that every separable conjugate space satisfying the approximation property is a factor of a space with a basis. Combining our results with those of Lindenstrauss [8] and a result of Pełczyński and Wojtaszczyk [13], we obtain the following

COROLLARY 4.12 (a) *There exists an absolute constant* $K (\leqq 16^{12})$ *such that f B is a finite dimensional Banach space, then there exists a finite dimensional Banach space H such that* $B \oplus H$ *has a basis with constant less than or equal to* K.

(*b*) *A separable Banach space satisfies the b.a.p. if and only if it is isomorphic to a complemented subspace of a space with a basis.*

PROOF. Our proof of Theorem 4.9 yields that there is a function g of two real variables, valued in the positive reals, such that if a Banach space X has a basis with constant b and X^* satisfies the β-m.a.p. and is separable, then X^* has a basis with constant less than or equal to $g(b,\beta)$. (A crude upper estimate for g is $g(b,\beta) \leqq 2bM^2M_0$, where $M = 2b^2 + 4\beta + 16b^2\beta$ and $M_0 = 3(1+2b^2)(1+M)$).

Now it follows immediately from the results of [8] that if A is a separable Banach space, then there exists a Banach space Z so that Z^{**} has a basis with constant less than or equal to 2, and $d(Z^{**}/Z, A) \leqq 2$; choose Z satisfying these conditions for $A = B^*$.

Then Z^{**} and hence Z^* satisfy the 2-m.a.p. by Lemma 3.1. Regarding $Z \subset Z^{**}$ and $Z^* \subset Z^{***}$, as is well known there is a (unique) projection P from Z^{***} onto Z^* with kernel $Z^\perp$, and $\|P\| = 1$. Then $d(Z^\perp, B) \leqq 2$, and in particular $Z^\perp$ is finite dimensional. It follows that Z^{***} satisfies the 4-m.a.p. and hence Z^{***} has a basis (b_n) with constant less than or equal to $\lambda = g(2,4)$. By Lemma 2.4, there is a subspace G of Z^{***} with $Z^\perp \subset G$ and $d(G, \operatorname{span}\{b_i\}_{i=1}^m) \leqq 2$, where $m = \dim G$. Letting $H = PG$, we have that $d(Z^\perp \oplus H, G) \leqq 4$, and hence $d(B \oplus H, \operatorname{span}\{b_i\}_{i=1}^m) \leqq 16$. Thus $B \oplus H$ has a basis with constant less than or equal to $K = 16\lambda$. (A crude upper bound for λ is 16^{11}.) Thus (a) is proved.

Turning to (b), we have that the "if" part is immediate, so we prove the only if assertion. Let X satisfy the b.a.p. By Theorem 1.1 of [13], there is a space Y with a f.d.d. $\{B_n\}_{n=1}^\infty$ such that X is isomorphic to a complemented subspace of Y. For each n choose H_n finite dimensional such that $B_n \oplus H_n$ has a basis with constant $\leqq K$. Now put $D = (\Sigma_{n=1}^{\infty} \oplus H_n)_2$; then $\{B_n \oplus H_n\}_{n=1}^{\infty}$ is a f.d.d. for $Y \oplus D$. Thus by Lemma 2.2 $Y \oplus D$ has a basis, and of course X is isomorphic to a complemented subspace of $Y \oplus D$. Q.E.D.

REMARK 4.13. It is shown in [12] that there exists a Banach space U such that U has a basis and every space with a basis is isomorphic to a complemented subspace of U. Thus Corollary 4.12 shows that every Banach space with the b.a.p. is isomorphic to a complemented subspace of U, from which it is easily seen that any space V which has the b.a.p. and is complementably universal for spaces having the b.a.p., is isomorphic to U. (This extends some of the results of [13].)

5. An application to $\mathscr{L}_p$ spaces

Let $1 \leqq p \leqq \infty$; then we denote by l_p^n the space of all n-tuples $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of numbers with $\| \alpha \| = (\Sigma_{i=1}^{n} | \alpha_i |^p)^{1/p}$ if $1 \leqq p < \infty$ and $\| \alpha \| = \max_{1 \leqq i \leqq n} | \alpha_i |$ if $p = \infty$. A Banach space X is said to be an $\mathscr{L}_{p,\lambda}$ space (cf. [9] p. 283) if for every finite dimensional subspace E of X there is a finite dimensional subspace F of X such that $E \subset F$ and $d(F, l_p^n) \leqq \lambda$, where $n = \dim F$. A Banach space is called an $\mathscr{L}_p$ space if it is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda < \infty$.

THEOREM 5.1. *Let $1 \leqq p \leqq \infty$ and let X be a separable $\mathscr{L}_p$ space. Then X has a basis.*

PROOF. We may and shall assume that X is infinite dimensional.

We consider first the case $1 \leqq p < \infty$. It is known (see [10] Th. III) that every $\mathscr{L}_p$ space X is a π space and that X^* is an $\mathscr{L}_q$ space, with $p^{-1} + q^{-1} = 1$ and $1 \leqq p \leqq \infty$. It follows from Theorem 1.3 that X has a finite dimensional decomposition determined by the projections $\{Q_n\}$. Put $P_n = Q_n - Q_{n-1}$, (with Q_0 denoting the zero operator); then since X is an $\mathscr{L}_{p,\lambda}$ space for some λ, for each n there is a finite dimensional subspace $F_n \subset X$ such that $E_n = P_n(X) \subsetneqq F_n$ and $d\{F_n, l_p^{d(n)}\} \leqq \lambda$, with $d(n) = \dim F_n$. Let $c = \sup_n \| P_n \|$,

$$G_n = (I_{F_n} - P_n)(F_n) \text{ and } G = (\sum_{n=1}^{\infty} \oplus G_n)_p;$$

then the fact that $\sup_n \| I - P_n \| \leqq 1 + c$ easily implies that G is isomorphic to a complemented subspace of l_p, and hence, by [11], G is isomorphic to l_p. It follows from [9] (p. 311 Proposition 7.3) that if $1 \leqq p < \infty$ then X has a complemented subspace isomorphic to l_p. Let H be a subspace of X such that $H \oplus l_p \approx X$; then because $l_p \oplus l_p \approx l_p$, we get that $X \approx l_p \oplus H \approx (l_p \oplus l_p) \oplus H \approx l_p \oplus (l_p \oplus H) \approx l_p \oplus X \approx G \oplus X$. Let U_n be the natural projection from $G = (\sum_1^\infty \oplus G_n)_p$ onto the nth component G_n and define $S_n: G \oplus X \to G \oplus X$ by $S_n(g+x) = U_n g + P_{nx}$ for every $g + x \varepsilon\ G \oplus X$. Then it is easy to see that $\{\sum_{j=1}^n S_j\}_{n=1}^\infty$ defines a finite dimensional decomposition in $G \oplus X$ and $S_n(G \oplus X) = G_n \oplus E_n$. Noting that $\sup_n d\{G_n \oplus E_n, F_n\} = b < \infty$ we get that for each n,

$$d\{G_n \oplus E_n, l_p^n\} \leqq d\{G_n \oplus E_n, F_n\} \cdot d\{F_n, l_p^{d(n)}\} \leqq b\lambda .$$

But $l_p^{d(n)}$ has a basis with constant 1, therefore, by Lemma 2.1(a) each space $G_n \oplus E_n$ has a basis with constant $\leqq b\lambda$. It follows from Lemma 2.2 that $X \approx G \oplus X$ has a basis. This proves the theorem for $1 \leqq p < \infty$.

Assume now that $p = \infty$. We still can construct the finite dimensional decomposition for X. Since X is an $\mathscr{L}_{\infty,\lambda}$ space, for each finite dimensional subspace $E \subset X$ there is a finite dimensional subspace $E \subset F$ and a projection P from X onto F such that $d\{F, l_\infty^{\dim(F)}\} \leqq 2\lambda$ and $\| P \| \leqq 2\lambda$. By Remark 4.6 the projections Q_n of the decomposition of X can be chosen such that $d\{Q_k(X), l_\infty^{d(k)}\} < 3\lambda$ where $d(k) = \dim Q_k(X)$. It follows from Lemma 2.1 that $\sup_k d\{Q_k^*(X^*), l_1^{d(k)}\} < \infty$ and therefore the space $Y = \operatorname{span}\{Q_k^*(X^*)\}_{k=1}^\infty$ is an $\mathscr{L}_1$ space. The (already proved) first part ($1 \leqq p < \infty$) implies that Y has a basis, and since $Q_k^* y \to y$ for all $y \in Y$ we get by Theorem 4.9 that X has a basis. This completes the proof of Theorem 5.1.

REMARK 5.2. Let $1 \leqq p \leqq \infty$ and let X be a separable infinite dimensional $\mathscr{L}_p$ space. By Remark 4.6, there is a f.d.d. for X with natural projections $\{Q_n\}$ such that $\sup_n d(Q_n(X), l_p^{d(n)}) < \infty$, where $d(n) = \dim Q_n(X)$. Although we know by 5.1 that X has basis, we do not know in general (if $p \neq 2$) that there is a basis for X with natural projections (Q_n) such that $\sup_n d(Q_n(X), l_p^n) < \infty$.

Appendix. To prove that $V \cap \tilde{L} \subset \widetilde{V \cap L}$ (see the notations in Section 3), let $y \in V \cap \tilde{L}$, let H be a finite dimensional subspace of Y^* such that $L = \{x \in Y: h(x) = 0 \text{ for all } h \in H\}$ and let D be the family of finite dimensional subspaces G of Y^* with $G \supset H$, directed by inclusion. For each $G \in D$ define $T: Y \to G^*$ by $(Tx)(g) = g(x)$ for all $x \in Y$ and $g \in G$. Then as noted at the beginning of the proof of Th. 3.1 of [10], $T^{**}(V) = T(V)$. Hence we may choose a $v_G \in V$ such that $g(v_G) = y(g)$ for all $g \in G$; then also $v_G \in L$ since $y(b) = 0$

for all $h \in H$. Thus the net $\{v_G\}_{G \in D}$ converges to y in the weak* topology, and has its range in $V \cap L$, whence $y \in \widetilde{V \cap L}$.

Added in proof. Corollary 4.12 and its consequences mentioned in Remark 4.13 have been independently discovered by A. Pełczyński. His proof yields a much better estimate on the constant K of 4.12 (see A. Pełczyński, "Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis", to appear, *Studia Math.*).

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University of Houston, Houston, Texas

University of California, Berkeley, California